Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(01)00045-4

J. Appl. Maths Mechs, Vol. 65, No. 3, pp. 395–410, 2001

0021-8928/01/$—see front matter

A DYNAMIC MODEL OF R AND D INVESTMENT†

C. WATANABE, S. A. RESHMIN and A. M. TARAS'YEV

Moscow

e-mail: reshmin@ipmnet.ru

(*Received* 19 *October* 2000)

A two-dimensional dynamic model of an investment process, which describes the change in the volume of production and the amount of accumulated investments, is considered. The amount of current investment serves as a control. A utility function in the form of an integral over a semi-infinite interval of the logarithm of the consumption index with a discounting factor is chosen as the criterion. The Pontryagin maximum principle is used to solve the problem. The first integral is found for the system of initial and associated equations; after which the system is reduced to a third-order system with a block structure. The equilibrium position of the reduced system is found. It is proved that it is a saddle-point. The trajectories in the neighbourhood of the equilibrium position are investigated qualitatively and the quasi-optimal control is constructed in the form of a synthesis. A numerical procedure is proposed for constructing the quasi-optimal control. An estimate of the calculation of the maximum value of the criterion is obtained. A comparison of the data obtained using the proposed model with aggregated data for the Japanese economy for the years from 1960 to 1992 is presented.

The dynamic model of an investment process considered here was proposed earlier in [1]. The model takes account of two fundamental trends: on the one hand, technology (accumulated annual investments) stimulates a production growth and, on the other hand, current investments consume part of the resources from the manufacturing sector. The problem involves finding the optimal investment which permits the maintenance of a balance between these two trends, the first of which ensures the effect of economic growth and the second is a risk factor. The effectiveness of the investments is characterized by an integral utility function which depends on the basic econometric indices: production, technology and current investments in the development of technology.

The problem under consideration is a classical problem of optimal economic growth [2–5]. The theory previously developed in [4] is used in analysing the model. A generalized model of economic growth for countries able to import technology has been investigated in [6].

Unlike in the case of the above-mentioned models, the optimal dynamics for growth in production are investigated as a function of current investments in the development of technology. Note that the effect of accumulated technology on growth in production has been studied previously in [7]. Results obtained earlier in [3, 8, 9] have also been used in setting up the model.

In this paper, a numerical algorithm is derived for finding the optimal trajectories of the model with any accuracy specified in advance.

1. DESCRIPTION OF THE MODEL

A mathematical model of the growth in production and technology [1] is considered which is described by the system of differential equations

$$\frac{\dot{y}(t)}{y(t)} = f_1 + f_2\left(\frac{T(t)}{y(t)}\right)^{\gamma} - g\frac{r(t)}{y(t)} \tag{1.1}$$

$$\dot{T}(t) = \frac{r(t-m) - \sigma T(t)}{1-\sigma} \tag{1.2}$$

Here $y(t)$ is the total yearly output (the overall volume of production of an industry or the production of an individual branch such as, for example, the processing industry), $T(t)$ is the technology which has

†*Prikl. Mat. Mekh.* Vol. 65, No. 3, pp. 408–425, 2001.

been accumulated in the industry (it can be measured in monetary units since technology can be bought and sold), $r(t)$ is the current annual amount of investment in the development of technology, f_1 is the endogenic rate of growth in production (the rate of growth of the industry without the accumulation of new technologies), $f_2(T(t)/y(t))^\gamma$ is the increment in the rate of growth of production due to the accumulation of new technologies, $gr(t)/y(t)$ is a factor for the slowdown in the rate of growth of production due to the withdrawal of credits from the production sector, m is the aggregated time of the commercialization of the developments and σ is the coefficient of "obsolescence" of the accumulated technologies.

Differential equation (1.2) is the continuous analogue of the finite-difference formula [10]

$$T(t) = r(t-m) + (1-\sigma)T(t-1)$$

In a simplified analysis, the parameters σ and m can be neglected and equated to zero. In this case, it can be assumed that

$$\dot{T}(t) = r(t) \tag{1.3}$$

The functions $f_1(t), f_2(t), g(t)$ can be expressed in terms of fundamental macroeconomic quantities [1]. We shall henceforth assume that the quantities f_1, f_2, g are constant.

The production $y(t)$ and the technology $T(t)$ are the fundamental variables of the model. The current investments $r(t)$ in the development of technology are not fixed in advance. It is required to find the optimal investment law $r(t)$ as a function of time. The utility function U_{t0}, represented in the form of an integral with a discount coefficient ρ [2–4]

$$W_{t_0} = \int_{t_0}^{\infty} e^{-\rho(t-t_0)} \ln D(t)dt \tag{1.4}$$

is adopted as the criterion of optimality. Here, $D(t)$ is consumption index, t_0 is the current time, t_0 is the initial instant of time, and the infinite upper limit of the integral denotes that a long-term development perspective is being considered.

We will specify the consumption index in the form [4]

$$D = D(t) = \left(\int_0^n x^\alpha(j)dj\right)^{1/\alpha}, \quad n = n(t) \tag{1.5}$$

where $x(j)$ is the amount of invented industrial goods of type j, n is the number of available (invented) types of industrial goods, $0 < \alpha < 1$ is a constant elasticity parameter and $1/(1-\alpha) > 1$ is a constant for the elasticity of substitution between two different types of goods.

We will now make the simplifying assumption that the quantities $x(j)$ are identical for the different indices j, that is

$$x(j) = y/n, \; y = y(t), \; n = n(t) \tag{1.6}$$

and that the number of invented products n depends on the accumulated investments T and the current change in technology r as follows

$$n = n(t) = be^{\varkappa t}T^{\beta_1}r^{\beta_2}, \quad T = T(t), \quad r = r(t) \tag{1.7}$$

Relations (1.6) and (1.7) show that n depends on the current activity of the investors, which is characterized by the value of r, and on the amount of investment which has been accumulated in the past in the development of the technology T. Moreover, an innovation process n has a tendency to fade out, which can be expressed by a factor $e^{\varkappa t}$.

Combining Eqs (1.4)–(1.7), we obtain the following expression for the utility function

$$W_{t_0} = U_{t_0} + A\int_{t_0}^{\infty} e^{-\rho(t-t_0)}(\varkappa t + \ln b)dt$$

Here

$$U_{t_0} = \int_{t_0}^{\infty} e^{-\rho(t-t_0)}(\ln y(t) + a_1 \ln T(t) + a_2 \ln r(t))dt \tag{1.8}$$

$$a_1 = A\beta_1,\ a_2 = A\beta_2,\ A = 1/\alpha - 1$$

The second term in the expression for the utility function W_{t0} is independent of the basic variables of the model y, T and r, and, consequently, one can henceforth consider the utility function U_{t0} instead of the utility function W_{t0}, bearing in mind that

$$W_{t_0} = U_{t_0} + A\rho^{-1}[\varkappa(t_0 + \rho^{-1}) + \ln b]$$

The structure of the utility function U_{t0} (1.8) shows that investors (the government or financial groups) are interested in growth in production y and, at the same time, in growth in the volume of accumulated technology T and the current change in its r value (the invention of new goods, etc.)

2. APPLICATION OF THE MAXIMUM PRINCIPLE

The control problem consists of finding the level of growth of technology $r^0 = r^0(t)$ in the class of piecewise-continuous functions $r(t)$ to which the optimal production $y^0 = y^0(t)$ and the optimal accumulation of technology $T^0 = T^0(t)$, that satisfy Eqs (1.1) and (1.3) and maximize utility function (1.8), correspond.

Problem (1.1), (1.3), (1.8) is a classical problem in the theory of optimal control. The Pontryagin maximum principle [11] can be used to solve it. Applications of this optimality principle to problems of economic growth have been considered previously [2–4].

We set up the Hamiltonian of problem (1.1), (1.3), (1.8)

$$H(y,T,r,\psi_1,\psi_2) = \ln y + a_1 \ln T + a_2 \ln r + \psi_1(f_1 y + f_2 T^{\gamma} y^{1-\gamma} - gr) + \psi_2 r \tag{2.1}$$

Calculating the maximum of Hamiltonian (2.1) with respect to the parameter r, we find that the maximum value is attained at the optimal rate of growth of the technology

$$r^0 = a_2(g\psi_1 - \psi_2)^{-1} \tag{2.2}$$

The dynamics of the associated variables ψ_1, ψ_2 are described by the system of equations

$$\dot{\psi}_1 = \rho\psi_1 - \frac{\partial H}{\partial y} = \rho\psi_1 - \frac{1}{y} - (1-\gamma)\psi_1 f_2 T^{\gamma}\frac{1}{y^{\gamma}} - \psi_1 f_1 \tag{2.3}$$

$$\dot{\psi}_2 = \rho\psi_2 - \frac{\partial H}{\partial T} = \rho\psi_2 - a_1\frac{1}{T} - \gamma\psi_1 f_2 y^{1-\gamma}\frac{1}{T^{1-\gamma}}$$

Combining Eqs (1.1), (1.3), (2.2) and (2.3), we obtain the closed system

$$\frac{\dot{y}}{y} = f_1 + f_2\left(\frac{T}{y}\right)^{\gamma} - \frac{ga_2}{(g\psi_1 - \psi_2)y},\quad \dot{T} = \frac{a_2}{g\psi_1 - \psi_2}$$

$$\frac{\dot{\psi}_1}{\psi_1} = \rho - \frac{1}{\psi_1 y} - (1-\gamma)f_2\left(\frac{T}{y}\right)^{\gamma} - f_1 \tag{2.4}$$

$$\frac{\dot{\psi}_2}{\psi_2} = \rho - a_1\frac{1}{\psi_2 T} - \gamma f_2\frac{\psi_1 y}{\psi_2 T}\left(\frac{T}{y}\right)^{\gamma}$$

It is required to find the solution of system (2.4) which satisfies the transversality condition of the maximum principle

$$\lim_{t\to\infty} e^{-\rho t} z(t) = 0 \tag{2.5}$$

where the function z is defined by the relation

$$z = \psi_1 y + \psi_2 T \tag{2.6}$$

The first integral (3.1) is found for system (2.4) in Section 3, and, because of this, an equivalent reduced system with separable variables $-x_1 = y/T, x_2 = \psi_1 y$ in one block and $x_3 = 1/T$ in the other, is obtained. Conditions are presented, under which a unique equilibrium position

$$x^0 = (x_1^0, x_2^0, x_3^0),\ x_1^0 > 0,\ x_2^0 > 0,\ x_3^0 = 0$$

exists in the reduced system.

An estimate of the eigenvalues and eigenvectors of the linearized system is then produced and it is shown that the position x^0 is a saddle point. In this case, trajectories of the reduced system exist which bring it to equilibrium. Consequently, a solution of system (2.4) exists which satisfies the transversality condition (2.5), (2.6).

In the general case, the optimal control r^0 has a complex structure. A quasi-optimal synthesis is proposed in Section 5 which is applicable in a small neighbourhood of the position x_1^0, x_3^0 and ensures the required behaviour of system (1.1), (1.3) (the same as under the optimal conditions)

$$\lim_{t\to\infty} y(t)/T(t) = x_1^0,\ \lim_{t\to\infty} 1/T(t) = x_3^0$$

In Section 6, a numerical algorithm is proposed for calculating the trajectories which lead to the reduced system (3.5) in the neighbourhood of the equilibrium position x^0. As a result, the corresponding programmed control for the initial system is obtained. This control (calculated in a finite time interval) can be used in combination with the quasi-optimal synthesis proposed in Section 5 which acts in an infinite interval.

An estimate of the error in calculating the maximum value of functional (1.8), which arises when the proposed method of control is used, is derived in Section 7 and it is shown that this error can be made as small as desired by an appropriate choice of the size of the neighbourhood of the equilibrium position x^0.

In Section 8, the results of a comparison of the optimal curves, calculated for certain model parameters, with actual econometric data are presented.

3. THE EXISTENCE OF EQUILIBRIUM AND OF THE OPTIMAL SOLUTION

Assertion 3.1. System (2.4), which describes the optimal dynamics, has a first integral

$$z = \psi_1 y + \psi_2 T = p^0 = (a_1 + a_2 + 1)/\rho \tag{3.1}$$

Proof. By virtue of system (2.4), differentiating function (2.6) we obtain a differential equation, the general solution of which is given by the formula

$$z(t) = Ce^{\rho t} + p^0 \tag{3.2}$$

The unique solution of the type (3.2) which satisfies transversality condition (2.5) is the constant function $z = p^0$, that is, when the constant C in (3.2) is equal to zero and there is no exponential part.

We now make the change of variables

$$x_1 = y/T,\ x_2 = \psi_1 y,\ x_3 = 1/T \tag{3.3}$$

and introduce the notation

$$u(x_1, x_2) = \frac{r^0}{y} = \frac{a_2}{(x_1 + g)x_2 - p^0 x_1},\quad u(x_1, x_2) > 0 \tag{3.4}$$

Taking account of the first integral (3.1), we reduce initial system (2.4) to a third-order system with a block structure

$$\begin{aligned} \dot{x}_1 &= f_1 x_1 + f_2 x_1^{1-\gamma} - (x_1 + g)x_1 u = F_1(x_1, x_2) \\ \dot{x}_2 &= \rho x_2 + \gamma f_2 x_2 x_1^{-\gamma} - 1 - g x_2 u = F_2(x_1, x_2) \\ \dot{x}_3 &= -x_1 x_3 u = F_3(x_1, x_2, x_3) \end{aligned} \tag{3.5}$$

In the subsequent analysis, we shall assume that the inequalities

$$0 \leq \gamma \leq 1, f_1 - \rho = \nu > 0 \tag{3.6}$$

are satisfied.

The first condition of (3.6) denotes the moderate effect of the accumulated technology T on the rate of growth of production $\dot{y}$. The second condition of (3.6) indicates that the endogenic rate of growth of production f_1 is strictly greater than the discount coefficient ρ.

It can be shown (see [1]) that system (3.5) only has a stationary point $x^0 = (x_1^0, x_2^0, x_3^0)$ under these conditions.

Assertion 3.2. Suppose conditions (3.6) are satisfied. Then, system (3.5) has stationary points x^0 which possess the following properties

$$\begin{aligned} &0 < r_1 < x_1^0 \leq z_1, \ 0 \leq r_2 < x_2^0 \leq z_2 \\ &(x_1^0 + g)x_2^0 - p^0 x_1^0 > 0, \ x_3^0 = 0 \end{aligned} \tag{3.7}$$

The parameters r_1, z_1 are the unique positive solutions of the corresponding equations

$$\frac{g}{r_1 + g} = \frac{\rho r_1^\gamma + \gamma f_2}{f_1 r_1^\gamma + f_2}, \quad \frac{p^0 g}{z_1 + g} = \frac{a_2 z_1^\gamma}{f_1 z_1^\gamma + f_2} \tag{3.8}$$

The parameters r_2 and z_2 are defined by the relations

$$r_2 = p^0 \min\left\{1 - \gamma, \ 1 - \frac{a_1}{f_1 p^0 + 1}\right\}, \ z_2 = p^0 \tag{3.9}$$

If the parameters of the model γ and f_2 are sufficiently small, that is,

$$f_2 \gamma^2 \leq \frac{a_2}{p^0} \min\left\{1, \frac{g(a_1 + 1)}{a_2}\right\} \tag{3.10}$$

then the point x^0 is unique.

4. A QUALITATIVE ANALYSIS OF THE EQUILIBRIUM POSITION

In order to establish the properties of the optimal control r^0 (2.2), we will investigate the stability of the equilibrium position x^0. We will calculate the Jacobi matrix $DF = \{F_{i,j}\}$ ($F_{i,j} = \partial F_i/\partial x_j$, $i,j = 1, 2, 3$) of the right-hand sides of system (3.5). For the partial derivatives $F_{i,j}$, we obtain the expressions

$$\begin{aligned} F_{1,1} &= f_1 + (1-\gamma) f_2 x_1^{-\gamma} - x_1 u + a_2^{-1} g x_2 (x_1 + g) u^2 \\ F_{1,2} &= a_2^{-1}(x_1 + g)^2 x_1 u^2, \ F_{1,3} = 0 \\ F_{2,1} &= -\gamma^2 f_2 x_2 x_1^{-(1+\gamma)} - a_2^{-1} g x_2 (p^0 - x_2) u^2 \\ F_{2,2} &= \rho + \gamma f_2 x_1^{-\gamma} + a_2^{-1} g p^0 x_1 u^2, \ F_{2,3} = 0 \\ F_{3,1} &= -a_2^{-1} g x_2 x_3 u^2, \ F_{3,2} = a_2^{-1}(x_1 + g) x_1 x_3 u^2, \ F_{3,3} = -x_1 u \end{aligned} \tag{4.1}$$

We will determine the signs of the coefficients $F_{i,j}$.

Assertion 4.1. The coefficients $F_{i,j}$ of the Jacobi matrix DF at the equilibrium position x^0 are defined by the expressions ($F^0_{i,j} = F_{i,j}(x^0)$)

$$\begin{aligned}
F^0_{1,1} &= -\gamma f_2 x_1^{-\gamma} - \frac{gp^0 x_1 (f_1 x_1^\gamma + f_2)^2}{a_2 x_1^{2\gamma}(x_1+g)^2} < 0\\
F^0_{1,2} &= \frac{x_1(f_1 x_1^\gamma + f_2)^2}{a_2 x_1^{2\gamma}} > 0,\quad F^0_{1,3} = 0\\
F^0_{2,1} &= -\gamma^2 f_2 x_2 x_1^{-(1+\gamma)} - \frac{g x_2 (p^0 - x_2)(f_1 x_1^\gamma + f_2)^2}{a_2 x_1^{2\gamma}(x_1+g)^2} < 0\\
F^0_{2,2} &= \rho + \gamma f_2 x_1^{-\gamma} + \frac{gp^0 x_1 (f_1 x_1^\gamma + f_2)^2}{a_2 x_1^{2\gamma}(x_1+g)^2} > 0\\
F^0_{2,3} &= F^0_{3,1} = F^0_{3,2} = 0\\
F^0_{3,3} &= -\frac{x_1^{1-\gamma}(f_1 x_1^\gamma + f_2)}{x_1+g} < 0
\end{aligned} \tag{4.2}$$

Here, $x_i = x_i^0 (i = 1, 2, 3)$.

Proof. We express u from the first equation of system (3.5) when $\dot{x}_1 = 0$ and substitute it into the expressions for the partial derivatives (4.1). Taking into account the first inequality of (3.7) and the second equality of (3.9), we obtain relations (4.2).

Using Assertion 4.1, we will now formulate the following assertion concerning the eigenvalues of the Jacobi matrix DF.

Assertion 4.2. The Jacobi matrix DF has just one eigenvalue with a positive real part, and, consequently, the equilibrium position x^0 is unstable.

Proof. Consider the second-order square matrix

$$D = \begin{Vmatrix} F^0_{1,1} & F^0_{1,2} \\ F^0_{2,1} & F^0_{2,2} \end{Vmatrix} \tag{4.3}$$

The partitioned structure of the matrix DF with the elements (4.2) indicates that, at the equilibrium position x^0, the eigenvalues of the matrix D are the eigenvalues of the matrix DF.

In accordance with relations (4.2), the trace of the matrix D is positive. It follows from the corresponding inequality that just one eigenvalue of the matrix D, and, consequently, of the matrix DF, has a positive real part.

We shall next assume that the discriminant DI of matrix D is negative

$$\mathrm{DI} = F^0_{1,1}F^0_{2,2} - F^0_{1,2}F^0_{2,1} < 0 \tag{4.4}$$

The constraints on the parameters of the system under consideration, subject to which inequality (4.4) is satisfied, have been found previously [1].

Assertion 4.3. Suppose inequality (4.4) is satisfied. Then, the Jacobi matrix DF has real eigenvalues: one of which is positive and two of which are negative. Consequently, the equilibrium position x^0 is a saddle.

Proof. Since the matrix DF has a partitioned structure and contains a negative diagonal element $F^0_{3,3}$, at least one of its eigenvalues

$$\mu_3 = -\frac{x_1^{1-\gamma}(f_1 x_1^\gamma + f_2)}{x_1+g} \tag{4.5}$$

is real and negative and the corresponding eigenvector $h_3 = (0, 0, 1)$ is a unit vector. If inequality (4.4) is satisfied, one positive eigenvalue μ_1 and one negative eigenvalue μ_2 of the matrix D exist.

Assertion 4.4. The positive eigenvalue μ_1 satisfies the condition

$$\mu_1 = \frac{\rho}{2} + \left(\frac{\rho^2}{4} + |\,\mathrm{DI}\,|\right)^{1/2} > \rho \tag{4.6}$$

and the negative eigenvalue μ_2 can be represented in the form

$$\mu_2 = -(\mu_1 - \rho) < 0 \tag{4.7}$$

Assertion 4.5. The eigenvectors h_1, h_2, corresponding to the eigenvalues μ_1, μ_2, have the positive components

$$h_1 = \frac{1}{n_1}(b, a+\mu_1, 0), \quad n_1 = (b^2 + (a+\mu_1)^2)^{1/2} \tag{4.8}$$

$$h_2 = \frac{1}{n_2}(a+\mu_1, c, 0), \quad n_2 = (c^2 + (a+\mu_1)^2)^{1/2} \tag{4.9}$$

Here,

$$a = |F_{1,1}|, \; b = F_{1,2}, \; c = |F_{2,1}| \tag{4.10}$$

If the discriminant DI is negative, the angles of inclination

$$\varphi_i = \operatorname{arctg}\frac{h_i^2}{h_i^1}, \quad i = 1,2 \tag{4.11}$$

of the eigenvectors are related to one another by the inequalities

$$0 \leqslant \varphi_2 < \varphi_1 < \pi/2 \tag{4.12}$$

Proof. Substituting the positive eigenvalue μ_1 into the equation for finding the eigenvectors

$$\begin{Vmatrix} \mu + a & -b \\ c & \mu - (\rho + a) \end{Vmatrix} \begin{Vmatrix} h^1 \\ h^2 \end{Vmatrix} = 0 \tag{4.13}$$

and considering its first row with the components of the eigenvector h_1

$$(\mu_1 + a)h_1^1 - bh_1^2 = 0 \tag{4.14}$$

we obtain expression (4.8).

Similarly, substituting the negative eigenvalue μ_2 (4.7) into Eq. (4.13) and considering its second row with the components of the eigenvector h_2

$$ch_2^1 + (\mu_2 - (\rho + a))h_2^2 = ch_2^1 - (\mu_1 + a)h_2^2 = 0 \tag{4.15}$$

we obtain relation (4.9).

Noting that the discriminant DI is negative

$$\mathrm{DI} = -a(\rho + a) + bc < 0$$

we obtain the chain of inequalities

$$0 > -a(\rho + a) + bc > -(\rho + a)^2 + bc > -(\mu_1 + a)^2 + bc$$

The last inequality ensures the relation

$$\mathrm{tg}\varphi_1 = \frac{\mu_1 + a}{b} > \frac{c}{\mu_1 + a} = \mathrm{tg}\varphi_2 \tag{4.16}$$

from which the required relation for the angles $\varphi_i (i = 1,2)$ (4.12) follows.

We will now consider the system obtained by linearizing system (3.5) in the neighbourhood of the stationary point $x^0 = (x_1^0, x_2^0, 0)$

$$\begin{aligned} \dot{x}_1 &= F_{1,1}^0(x_1 - x_1^0) + F_{1,2}^0(x_2 - x_2^0) \\ \dot{x}_2 &= F_{2,1}^0(x_1 - x_1^0) + F_{2,2}^0(x_2 - x_2^0) \\ \dot{x}_3 &= F_{3,3}^0 x_3 \end{aligned} \tag{4.17}$$

Summing the properties of the Jacobi matrix, which are reflected in Assertions 4.1–4.5, the following properties of linear system (4.17) can be obtained.

Assertion 4.6. Suppose conditions (3.6), (3.10) and (4.4) are satisfied. Then, the linear system (4.17) has the following properties.

1. The equilibrium position x^0 is unique and is a saddle.
2. For any pair x_1^*, x_3^*, a unique value x_2^* exists such that the initial point $x^* = (x_1^*, x_2^*, x_3^*)$ lies in the plane generated by the eigenvectors h_2 h_3, which correspond to the negative eigenvalues μ_2, μ_3. The trajectory $x^*(\cdot)$ of linear system (4.17), which begins at the point x^*, tends to the equilibrium point x^0.
3. The second component $x_2(\cdot)$ of the other trajectories $x(\cdot)$, which begin at the points $x = (x_1^*, x_2, x_3^*), x_2 \neq x_2^*$, tend to infinity at the rate of an exponential function with a growth exponent $\mu_1 > \rho$

$$x_2(t) \to \infty,\ t \to \infty \tag{4.18}$$

Proof. Property 1 follows from Assertions 3.2 and 4.3. Property 2 follows from the fact that the first component h_2^1 of the eigenvector h_2 and the third component h_3^3 of the eigenvector h_3 are strictly positive, and, consequently, the plane generated by the first and third components x_1, x_3 can be orthogonally projected onto the plane generated by the eigenvectors h_2, h_3.

Property 3 follows from the instability of the equilibrium x^0 and the property of the positive eigenvalue $\mu_1 > \rho$ (4.6).

Properties 1–3 of linear system (4.17) are then used to investigate the optimal trajectories of non-linear system (3.5). According to the Grobman–Hartman theorem [12], non-linear system (3.5), as well as linear system (4.17), allow of trajectories which tend to the equilibrium x^0.

Assertion 4.7. Non-linear system (3.5) inherits the properties of the convergence of the solution to the equilibrium position, which are characteristic of linear system (4.17), that is, trajectories $x^0(\cdot)$ exist which bring non-linear system (3.5) from the initial state x^* to the equilibrium position x^0

$$\lim_{t \to \infty} x_i^0(t) = x_i^0,\ \ x_i^0(t_0) = x_i^*,\ \ i = 1,2,3 \tag{4.19}$$

Only such trajectories satisfy conditions (2.4) of the maximum principle and transversality conditions (2.5), and they are the optimal trajectories.

We will now describe the behaviour of the optimal trajectory $x^0(\cdot)$ in the neighbourhood of the equilibrium position x^0 of non-linear system (3.5). The third component $x_3^0(\cdot) = 1/T^0$ tends to zero $x_3^0 = 0$ (3.7) at a negative rate (3.5) and the magnitude of the accumulated technology $T^0 = T^0(t)$ therefore increases monotonically to infinity.

The first component $x_1^0(\cdot) = y^0/T^0$ tends to the positive equilibrium value $x_1^0 > 0$ (3.7). Consequently, the volume of production also grows to infinity with the same asymptotic growth index as the technology T^0 and, furthermore, its derivative (2.4) is strictly positive $\dot{y}^0(t) > 0$, $t \geq t_0$.

If the quantity $x_1^* = y^0(t_0)/T^0(t_0)$ exceeds the equilibrium value $x_1^0, x_1^0 \leq x_1^*$ at the beginning of the control process, the optimal ratio $x_1^0(t) = y^0(t)/T^0(t)$ decreases from the initial value x_1^* to the equilibrium value x_1^0. This means that the relative rate of growth of the accumulated technology T^0 is greater than the relative rate of growth of production y^0.

5. QUASI-OPTIMAL SYNTHESIS IN THE NEIGHBOURHOOD OF THE POINT (x_1^0, x_3^0)

It should be noted that the problem of finding the optimal trajectory $x^0(\cdot)$, which brings system (3.5) to the equilibrium position x^0, is quite difficult because of its instability. We will construct a quasi-optimal control in the form of a synthesis which brings the new coordinates $x_1(\cdot)$, $x_3(\cdot)$ of the system into the position (x_1^0, x_3^0). For this purpose, we will consider a linear variation of the second coordinate $x_2(\cdot)$ in the first and third equations of (3.5)

$$x_2 = x_2^0 + \omega(x_1 - x_1^0), \quad \omega \geq 0 \tag{5.1}$$

$$\dot{x}_1 = f_1 x_1 + f_2 x_1^{1-\gamma} - (x_1 + g)x_1 u^*(x_1), \quad \dot{x}_3 = -x_1 x_3 u^*(x_1) \tag{5.2}$$

$$x_1(t_0) = x_1^*, \quad x_3(t_0) = x_3^*$$

The function

$$u^*(x_1) = a_2(d + k(\omega)(x_1 - x_1^0) + \omega(x_1 - x_1^0)^2)^{-1}$$

is obtained when expression (5.1) is substituted into (3.4). The parameters d and k are defined by the relations

$$d = gx_2^0 - \left(p^0 - x_2^0\right)x_1^0, \quad k = k(\omega) = k_1\omega + k_2 \tag{5.3}$$

$$k_1 = x_1^0 + g, \quad k_2 = -\left(p^0 - x_2^0\right)$$

and the initial values x_1^*, x_3^* must satisfy the conditions

$$x_1^0 \leq x_1^* < x_1^0 + \bar{x}_1(\omega), \quad x_3^* > 0 \tag{5.4}$$

where

$$\bar{x}_1(\omega) = \begin{cases} 2d/\left(|k(\omega)| + \xi^{1/2}\right), & \xi \geq 0 \\ +\infty, & \xi < 0 \end{cases}, \quad \xi = k^2(\omega) - 4\omega d \tag{5.5}$$

Using the formulae for the new variables (3.3), we obtain the corresponding feedback control law $r = r(y, T)$ for initial system (1.1), (1.3)

$$r^* = \dot{T} = -x_3^{-2}\dot{x}_3 = x_1 x_3^{-1} u^*(x_1) = yu^*(y/T) \tag{5.6}$$

We will now formulate the conditions under which system (5.2) is stable.

Assertion 5.1. Suppose the coefficient ω in (5.1) satisfies the conditions

$$0 \leq \omega \leq \frac{gp^0}{\left(x_1^0 + g\right)^2} = \omega_1 \tag{5.7}$$

Then, the quasi-optimal synthesis r^* (5.6) transfers the trajectories $x^*(\cdot)$ of system (5.2) from the initial state x_1^*, x_3^* (5.4) into the equilibrium position x_1^0, x_3^0.

Proof. The convergence of $x_1^*(\cdot)$ to the equilibrium value x_1^0 follows from the property of asymptotic stability: the corresponding total derivative must have a negative sign

$$\frac{dF_1\left(x^0\right)}{dx_1} < 0 \tag{5.8}$$

Taking account of relations (4.2) for the partial derivatives $F^0_{1,i}$ and the linear dependence (5.1) of the coordinates x_i $(i = 1, 2)$, we obtain

$$\frac{dF_1(x^0)}{dx_1} = F^0_{1,1} + F^0_{1,2}\omega = -\gamma f_2 x_1^{-\gamma} - \frac{x_1\left(f_1 x_1^\gamma + f_2\right)^2}{a_2 x_1^{2\gamma}\left(x_1 + g\right)^2}\left(gp^0 - \left(x_1 + g\right)^2 \omega\right) \tag{5.9}$$

It is obvious that, in the case of conditions (5.7) $dF_1/dx_1 < 0$, which guarantees the required asymptotic stability.

The relations

$$F^0_{3,1} = 0, \quad F^0_{3,3} < 0 \tag{5.10}$$

are the conditions of asymptotic stability for the third equation of system (3.5) and guarantee that the third component $x_3^*(\cdot)$ vanishes.

We will now consider the natural situation when the tangent of the angle of inclination ω_0 of the eigenvector h_2 (4.9), corresponding to the negative eigenvalue μ_2 (4.7), of the Jacobi matrix D (4.3) is selected as the parameter ω in the control law (5.6).

$$\omega_0 = \frac{a + \mu_2}{b} = \frac{a + \rho - \mu_1}{b} = \frac{c}{a + \mu_1} \tag{5.11}$$

Assertion 5.2. The tangent of the angle of inclination ω_0 (5.11) of the eigenvector h_2 (4.9), which corresponds to the negative eigenvalue μ_2 (4.7), satisfies the relations

$$0 \leqslant \omega_0 < \omega_1 \tag{5.12}$$

and, consequently, the quasi-optimal synthesis $r^* = r^*(\omega_0)$ (5.6) with the coefficient ω_0 transfers the trajectories $x^*(\cdot)$ from the initial state x_1^*, x_3^* into the equilibrium state x_1^0, x_3^0.

Proof. Taking relation (4.6) into account, we obtain the following sequence of inequalities

$$\omega_1 = \frac{a}{b} > \frac{a}{b} - \frac{\mu_1 - \rho}{b} = \frac{a + \rho - \mu_1}{b} = \omega_0 = \frac{c}{a + \mu_1} \geqslant 0$$

Finally, we note certain properties of the quasi-optimal trajectories $x^*(\cdot)$ which bring the system to the equilibrium position x^0. A more complete investigation of the behaviour of the system under consideration using the quasi-optimal synthesis r^* (5.6) is given in [1].

Remarks. 1. Under quasi-optimal conditions the third component $x_3^*(\cdot) = 1/T^*$ tends to zero at a negative rate (5.10). It follows from this that the amount of accumulated technology T^* increases monotonically to infinity at the rate of an exponential function with an exponent $|\mu_3| > f_1 - \rho > 0$ (4.5).

The first component $x_1^*(\cdot) = y^*/T^*$ tends to a positive equilibrium value $x_1^0 > 0$. Hence, the volume of production y^* also tends to infinity with the same asymptotic growth index as the technology T^*.

If, at the start of the control process, the ratio x_1^* of production y^* to technology T^* exceeds the equilibrium value x_1^0, that is

$$x_1^0 \leqslant x_1^* < \frac{g x_2^0}{p^0 - x_2^0} \tag{5.13}$$

then the quasi-optimal ratio $x_1^*(\cdot) = y^*/T^*$ decreases from the initial value x_1^* to the equilibrium value x_1^0. This means that, in this case, the relative rate of growth of the accumulated technology T^* is greater than the relative rate of growth in production y^*.

2. In expression (5.6), the function $u^*(y/T)$ tends to a positive constant value when $t \to \infty$. Hence, the amount of investment under quasi-optimal conditions r^* (5.6) also tends to infinity with the same asymptotic growth index as the production y^* and the technology T^*.

6. A NUMERICAL ALGORITHM FOR FINDING QUASI-OPTIMAL CONTROL

We will not present a constructive numerical procedure for finding the quasi-optimal control which approaches the optimal control with any accuracy specified in advance. The corresponding error in calculating the functional is presented in Section 7. The process of finding the quasi-optimal control consists of two basic stages. The trajectory of reduced system (3.5), which falls into the neighbourhood of the equilibrium position x^0 (the trajectory is calculated over a finite time interval, is found during the first stage. The corresponding control and trajectory for the initial system are determined in accordance with (3.3). In the second stage, the synthesis of the control (5.6) (see Section 5) which transfers system (3.5) to the position x_1^0, x_3^0 is used.

We will assume that all the parameters of the model, the initial instant of time t_0 and the corresponding initial values of production $y(t_0)$ and technology $T(t_0)$ are fixed and, moreover, a sufficiently small parameter ε, which determines the accuracy of the algorithm, is specified. Then, the coordinates x_1^0, x_2^0 of the equilibrium position of system (3.5) are found from the system of equalities $F_1(x_1, x_2) = 0$, $F_2(x_1, x_2) = 0$.

We now consider the ε-neighbourhood of the point (x_1^0, x_2^0) in the plane (x_1, x_2) (Fig. 1). The behaviour of the variables x_1, x_2 will henceforth be investigated separately from x_3 since system (3.5) has a block structure. In the ε-neighbourhood, we fix a certain point (x_1^*, x_2^*), lying in a ray beginning at (x_1^0, x_2^0) and having an angle of inclination with a tangent ω which satisfies inequality (5.7). In particular, this ray can be parallel to the eigenvector which corresponds to the negative eigenvalue of the Jacobi matrix D.

We will now describe an algorithm for calculating the trajectory of system (3.5) which, at the instant of time $t^* > t_0$, passes through a point $x^* = (x_1^*, x_2^*, x_3^*)$, where the values of t^*, x_3^* are not fixed and will be determined later. This procedure can be developed in several steps. Initially, we consider the values $x_1 = x_1^*, x_2 = x_2^*$ as the initial values and integrate the system of the first two equations in (3.5) in reverse time until the equality $x_1(t) = x_1(t_0) = y(t_0)/T(t_0)$ turns out to be satisfied; this condition determines the time interval over which the calculation is carried out. As a result, we find the functions $x_1(t)$ and $x_2(t)$, $(t_0 \leq t \leq t^*)$. We then substitute $x_1(t)$ and $x_2(t)$ into the third equation in (3.5) and integrate it in direct time with the initial condition $x_3(t_0) = 1/T(t_0)$. As a result, we find the function $x_3(t)$ and the final value $x_3^* = x_3(t^*)$. In accordance with (3.3), we also obtain the values of the technology $T(t) = 1/x_3(t)$, the production $y(t) = x_1(t)/x_3(t)$ and the investment $r = \dot{T}$. Figure 1 illustrates the operation of the algorithm. The arrow directed to the right symbolizes integration in reverse time while the arrow directed to the left symbolizes integration in direct time.

We will now show that the coordinate x_1 always attains its initial value $x_1(t_0)$ during integration in reverse time if the parameters of the system satisfy the following constraints

$$x_1^0 < x_1(t_0) < \left(\frac{gf_1}{\gamma^2 f_2}\right)^{1/(1-\gamma)}, \quad x_2^0 > \frac{1}{f_1} \tag{6.1}$$

Conditions (6.1) are sufficient for the successful operation of the proposed algorithm.

According to Eqs (3.5), we have the following relations which will be used below

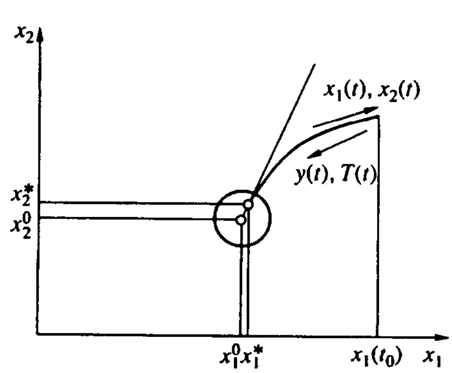

Fig. 1

$$\ddot{x}_1 = \frac{\dot{x}_1^2}{x_1} + x_1\left(\frac{\dot{x}_1}{x_1}\right)^{\cdot} = \frac{\dot{x}_1^2}{x_1} - \left(\gamma f_2 x_1^{-\gamma} + x_1 u\right)\dot{x}_1 - x_1(x_1 + g)\dot{u} \tag{6.2}$$

$$\ddot{x}_2 = \frac{\dot{x}_2^2}{x_2} + x_2\left(\frac{\dot{x}_2}{x_2}\right)^{\cdot} = \frac{\dot{x}_2^2}{x_2} - \gamma^2 f_2 x_2 x_1^{-(\gamma+1)}\dot{x}_1 - x_2 g\dot{u} + \frac{\dot{x}_2}{x_2} \tag{6.3}$$

$$\frac{\dot{x}_2}{x_2} = \frac{g}{x_1 + g}\frac{\dot{x}_1}{x_1} + F(x_1, x_2) \tag{6.4}$$

$$F(x_1, x_2) = -\frac{g\left(f_1 + f_2 x_1^{-\gamma}\right)}{x_1 + g} + \rho + \gamma f_2 x_1^{-\gamma} - \frac{1}{x_2}$$

We will initially make the assumption (it will be proved below) that, when calculating the quasi-optimal trajectory, conditions $\ddot{x}_1(t) > 0$ and $\ddot{x}_2(t) > 0$ are satisfied at all instants of time t, $t_0 \leqslant t \leqslant t^*$. It is then obvious that $\ddot{x}_1(t) < 0, \ddot{x}_2(t) > 0$ and that the coordinate x_1 attains its initial value $x_1(t_0)$. As a result, the numerical algorithm which is used finds the quasi-optimal trajectory.

We obtain an expression for the derivative of the function u with respect to time t

$$\dot{u} = -\frac{u^2}{a_2}\left[\left(x_2 - p^0\right)\dot{x}_1 + (x_1 + g)\dot{x}_2\right] \tag{6.5}$$

Using the condition $x_2 - p^0 > -gx_2/x_1$, which follows from (3.4), and equality (6.4), we obtain an estimate which is satisfied when calculating a trajectory which falls within the neighbourhood of the equilibrium position,

$$\dot{u} < -\frac{u^2}{a_2}\left[-gx_2\frac{\dot{x}_1}{x_1} + (x_1 + g)\dot{x}_2\right] =$$

$$= -\frac{u^2(x_1 + g)x_2}{a_2}\left[\frac{\dot{x}_2}{x_2} - \frac{g}{x_1 + g}\frac{\dot{x}_1}{x_1}\right] = -\frac{u^2(x_1 + g)x_2}{a_2}F(x_1, x_2) \tag{6.6}$$

We will now show that the function $F(x_1, x_2)$, introduced earlier in (6.4), is positive. We have

$$\frac{\partial F(x_1, x_2)}{\partial x_1} = \frac{g\gamma f_2 x_1^{-(1+\gamma)}}{x_1 + g} + \frac{g\left(f_1 + f_2 x_1^{-\gamma}\right)}{(x_1 + g)^2} - \gamma^2 f_2 x_1^{-(1+\gamma)} =$$

$$= \frac{gf_1 + f_2 x_1^{-(1+\gamma)}\left(-\gamma^2 x_1^2 + gx_1\left(1 + \gamma - 2\gamma^2\right) + g^2\gamma(1-\gamma)\right)}{(x_1 + g)^2} \geqslant$$

$$\geqslant \frac{gf_1 - \gamma^2 f_2 x_1^{1-\gamma}}{(x_1 + g)^2} > \frac{gf_1 - \gamma^2 f_2\left(x_1(t_0)\right)^{1-\gamma}}{(x_1 + g)^2} > 0$$

$$\frac{\partial F(x_1, x_2)}{\partial x_2} = \frac{1}{x_2^2} > 0$$

Hence,

$$\dot{F} = \frac{\partial F(x_1, x_2)}{\partial x_1}\dot{x}_1 + \frac{\partial F(x_1, x_2)}{\partial x_2}\dot{x}_2 < 0$$

Moreover, the equality $F(x_1^0, x_2^0) = 0$ is satisfied by virtue of (6.4). Hence, in the operation of the algorithm $F(x_1, x_2) > 0$ and, according to (6.6), $\dot{u} < 0$, and it follows from (6.3) that $\ddot{x}_1(t) > 0$.

We will now prove that $\ddot{x}_2(t) > 0$. Using the inequality

$$\frac{\dot{x}_2}{x_2} > \frac{g}{x_1+g}\frac{\dot{x}_1}{x_1}$$

which follows from (6.4) when $F(x_1, x_2) > 0$, and expression (6.3), we obtain

$$\ddot{x}_2 > \frac{\dot{x}_2^2}{x_2} + \left(\frac{g}{x_1+g} - \gamma^2 f_2 x_2 x_1^{-\gamma}\right)\frac{\dot{x}_1}{x_1} - x_2 g\dot{u} > 0 \tag{6.7}$$

Here, we have used the fact that, in the case of the previously introduced constraints (6.1), the expression in the brackets in (6.7) is negative

$$\frac{g}{x_1+g} - \gamma^2 f_2 x_2 x_1^{-\gamma} = x_1^{-\gamma}\left(\frac{g x_1^{\gamma}}{x_1+g} - \gamma^2 f_2 x_2\right) <$$

$$< x_1^{-\gamma}\left[g x_1(t_0)^{-(1-\gamma)} - \gamma^2 f_2 x_2^0\right] = x_1^{-\gamma}\left[g\left(\frac{\gamma^2 f_2}{g f_1}\right) - \gamma^2 f_2 x_2^0\right] < 0$$

Thus, the guaranteed operation of the numerical algorithm, subject to constraints (6.1), has been proved and it has been shown that the inequalities

$$\ddot{x}_1(t) > 0,\quad \dot{x}_1(t) < 0,\quad \ddot{x}_2(t) > 0,\quad \dot{x}_2(t) < 0,\quad \dot{u}(t) < 0$$

hold when $t_0 \leq t \leq t^*$.

7. AN ESTIMATE OF THE ACCURACY OF THE ALGORITHM

In this Section, an analytical estimate is obtained of the error in calculating the maximum value of the utility function, which arises as a result of the use of the proposed quasi-optimal control.

We substitute the expression for the optimal control law (2.2) into functional (1.8). Then, from (3.3) and (2.6), we obtain

$$U_{t_0} = \int_{t_0}^{\infty} e^{-\rho(t-t_0)}\left[(1+a_2)\ln x_1 + a_2 \ln u\right]dt + \eta \tag{7.1}$$

$$\eta = -p_0\rho\int_{t_0}^{\infty} e^{-\rho(t-t_0)} \ln x_3 dt$$

Taking account of the fact that

$$\lim_{t\to\infty}\frac{\ln x_3}{e^{\rho(t-t_0)}} = \lim_{t\to\infty}\frac{(\ln x_3)^{\cdot}}{\left(e^{\rho(t-t_0)}\right)^{\cdot}} = \lim_{t\to\infty}\frac{-x_1 u e^{-\rho(t-t_0)}}{\rho} = 0$$

we rewrite the last term in (7.1) in the form

$$\eta = p^0 e^{-\rho(t-t_0)}\ln x_3\Big|_{t_0}^{\infty} - p^0\int_{t_0}^{\infty} e^{-\rho(t-t_0)}\frac{\dot{x}_3}{x_3}dt =$$

$$= -p^0 \ln x_3(t_0) + p^0\int_{t_0}^{\infty} x_1 u e^{-\rho(t-t_0)}dt$$

It is now obvious that the error in calculating the maximum value of functional (7.1) is exclusively associated with the accuracy in determining the functions $x_1(t)$, $x_2(t)$ in the time intervals (t_0, t^*), (t^*, ∞). Note that, in formula (7.1), the upper limit of integration can be put equal to t^* and the value of the resulting integral can then be found numerically. The additional quantity

$$U_{t^*} = \int_{t^*}^{\infty} e^{-\rho(t-t_0)}\left[(1+a_2)\ln x_1 + a_2 \ln u + p^0 x_1 u\right]dt \tag{7.2}$$

which should make a contribution to integral (7.1) but has been dropped on changing the limit of integration, can be estimated. The question arises: how is the parameter ε, which specifies the dimensions of the neighbourhood of the equilibrium position, to be chosen if the permissible error Δ_{t_0} in calculating the functional U_{t0} is specified?

We next consider two additional problems.

Problem 1. It is required to determine the error in calculating the functions $x_1(t)$, $x_2(t)$ in a time interval (t_0, t^*) and the error in calculating functional (7.1) associated with this.

Solution. We initially estimate the duration $t_0 - t^*$ of the process of integration in reverse time. Using relations (3.5), (6.2) and (6.6), we obtain

$$\begin{aligned}\ddot{x}_1 &> -\left(\gamma f_2 x_1^{-\gamma} + x_1 u\right)\dot{x}_1 - x_1(x_1+g)\dot{u} = G(x_1,x_2) > G(x_1^*,x_2^*) = \\ &= \frac{\partial G(x_1^0,x_2^0)}{\partial x_1}\delta x_1(t^*) + \frac{\partial G(x_1^0,x_2^0)}{\partial x_2}\delta x_2(t^*) = C\varepsilon\end{aligned} \tag{7.3}$$

The notation

$$\begin{aligned}G(x_1,x_2) &= -\left(\gamma f_2 x_1^{1-\gamma} + x_1^2 u\right)\left(f_1 + f_2 x_1^{-\gamma} - (x_1+g)u\right) + \\ &+ x_1 \frac{u^2(x_1+g)^2 x_2}{a_2} F(x_1,x_2)\end{aligned}$$

has been introduced here.

Hence, we have the limit

$$t^* - t_0 < \left[\frac{2\left(x_1(t_0) - x_1^0\right)}{C\varepsilon}\right]^{1/2} \tag{7.4}$$

By virtue of the theorem on the continuous dependence of the solution on the initial data [13], integration of the first two equations of reduced system (3.5) in reverse time leads to an error of the order of $\varepsilon^{1/2}$ in determining $x_1(t)$, $x_2(t)$ in the interval $t_0 < t < t^*$ which, in turn, gives an error of $C_1\varepsilon^{1/2}(1 - e^{-\rho(t_0-t^*)})$ in calculating functional (7.1) (the immediate value of the constant C_1 is determined by the parameters of the system).

Problem 2. Suppose that, for any $t > t^* > t_0$, the functions $x_1(t)$, $x_2(t)$ can be represented in the form

$$x_1(t) = x_1^0 + \delta x_1(t), \quad x_2(t) = x_2^0 + \delta x_2(t) \tag{7.5}$$

Here, x_1^0, x_2^0 specify the equilibrium position of system (3.5) and the quantities $\delta x_1(t)$, $\delta x_2(t)$ are sufficiently small so that

$$\lim_{t\to\infty} x_1(t) = x_1^0, \quad \lim_{t\to\infty} x_2(t) = x_2^0$$

(these conditions are satisfied if $x_1(t)$ and $x_2(t)$ change in accordance with the optimal law or the quasi-optimal law (5.6)). It is required to estimate the value of functional (7.2).

Solution. We assume that the constraints

$$|\delta x_1| < \Delta_1 \sim \varepsilon, \quad |\delta x_2| < \Delta_2 \sim \varepsilon \tag{7.6}$$

where Δ_1, Δ_2 are certain constants, are satisfied for any $t > t^*$. We substitute expressions (7.5) into (7.2)

$$U_{t^*} = \frac{e^{-\rho(t^*-t_0)}}{\rho}\left[(1+a_2)\ln x_1^0 + a_2 \ln u\left(x_1^0, x_2^0\right) + p^0 x_1^0 u\left(x_1^0, x_2^0\right)\right] + \Delta \tag{7.7}$$

Here

$$\Delta = \int_{t^*}^{\infty} e^{-\rho(t-t_0)}\left[p^0 a_2^{-1} x_2^0 u^2\left(g\delta x_1 - \left(x_1^0 + g\right)\delta x_2\right) + \right.$$

$$\left. +(1+a_2)\frac{\delta x_1}{x_1^0} - a_2 u\left(\left(x_2^0 - p^0\right)\delta x_1 + \left(x_1^0 + g\right)\delta x_2\right)\right]dt + o(\varepsilon) \tag{7.8}$$

Using inequalities (7.6), we obtain

$$|\Delta| < \frac{e^{-\rho(t^*-t_0)}}{\rho}\left[p^0 a_2^{-1} x_2^0 u^2\left(g\Delta_1 + \left(x_1^0 + g\right)\Delta_2\right) + (1+a_2)\frac{\Delta_1}{x_1^0} + \right.$$

$$\left. + a_2 u\left(\left(p^0 - x_2^0\right)\Delta_1 + \left(x_1^0 + g\right)\Delta_2\right)\right] + o(\varepsilon) = C_2 \varepsilon e^{-\rho(t^*-t_0)} + o(\varepsilon) \tag{7.9}$$

According to relation (7.9), the absolute value of Δ becomes infinitesimal when $\varepsilon \to 0$. Hence, the following result has been obtained: if $\varepsilon \to 0$, then, in the case of the quasi-optimal control law (5.6), the value of the functional U_{t^*} tends to a constant, determined by the first term of expression (7.7).

Summing the estimates of the errors obtained in the solution of Problems 1 and 2, we obtain the following transcendental equation for ε

$$\Delta_{t_0} = C_1 \varepsilon^{1/2}\left(1 - e^{-\lambda/\varepsilon^{1/2}}\right) + C_2 \varepsilon e^{-\lambda/\varepsilon^{1/2}}, \quad \lambda = \frac{1}{\rho}\left[\frac{2\left(x_1(t_0) - x_1^0\right)}{C}\right]^{1/2}$$

where Δ_{t0} is the required accuracy in calculating the maximum value of utility function (1.8).

8. ECONOMETRIC IDENTIFICATION

The mathematical model being considered was tested on the data for the Japanese processing industry for the period from 1960 to 1992. According to material from the Tokyo Institute of Technology (see [10]) the Japanese economy developed under difficult conditions in the 1960's: rapid economic growth was accompanied by a labour shortage and an energy crisis. The process of economic development was irregular. At the beginning of the 1970's, strenuous efforts were made to stabilize the process and to control it. As a result, the economic development acquired a stable and controlled character. The policy of investment in the development of science and new technologies was one of the basic control levers.

The test results are presented below. Variation of the model parameters enables one to obtain clusters of theoretically optimal trajectories. By comparing these clusters of trajectories with the empirical data, it is possible to draw conclusions regarding the effectiveness of the control of the economy and to estimate the optimality of the investment for different economic development scenarios. The clusters which have been synthesized can also be used to determine the sensitivity of the optimal trajectories to variations in the various parameters of the model or combinations of them. Such a sensitivity analysis is particularly important when the process of economic development being considered is optimal or close to optimal. In this situation, the parameters, to a change in which the optimal trajectories are most sensitive, can be identified by experts. Control of these vital parameters is an important problem in macroeconomic management.

Clusters of synthesized optimal trajectories, obtained by varying the elasticity coefficient within the range $0.90 < \alpha < 0.94$, are shown in Fig. 2. For the time interval covering the years from 1970 to 1992, the clusters of optimal trajectories are compared with the empirical data for four main indices: $y(t)$ is the production (trillions of yen), $T(t)$ is the technology (trillions of yen), $y(t)/T(t)$ is the productivity of technology, $r(t-m)/y$ is the intensity of investment in the development of technology and t is the time (years). The optimal trajectory which is closest to the actual econometric data (denoted by the dashed curve) corresponds to the following set of model parameters: $\gamma = 1.0$, $\alpha = 0.92$, $\rho = 0.035$ ((year)$^{-1}$), $\beta_1 = 0.5$, $\beta_2 = 0.5$, $f_1 = 0.042$ ((year)$^{-1}$, $f_2 = 0.015$ ((year)$^{-1}$), $g = 0.6$ and the initial values $t_0 = 1970$, $y(t_0) = 1.504$, $T(t_0) = 0.1034$.

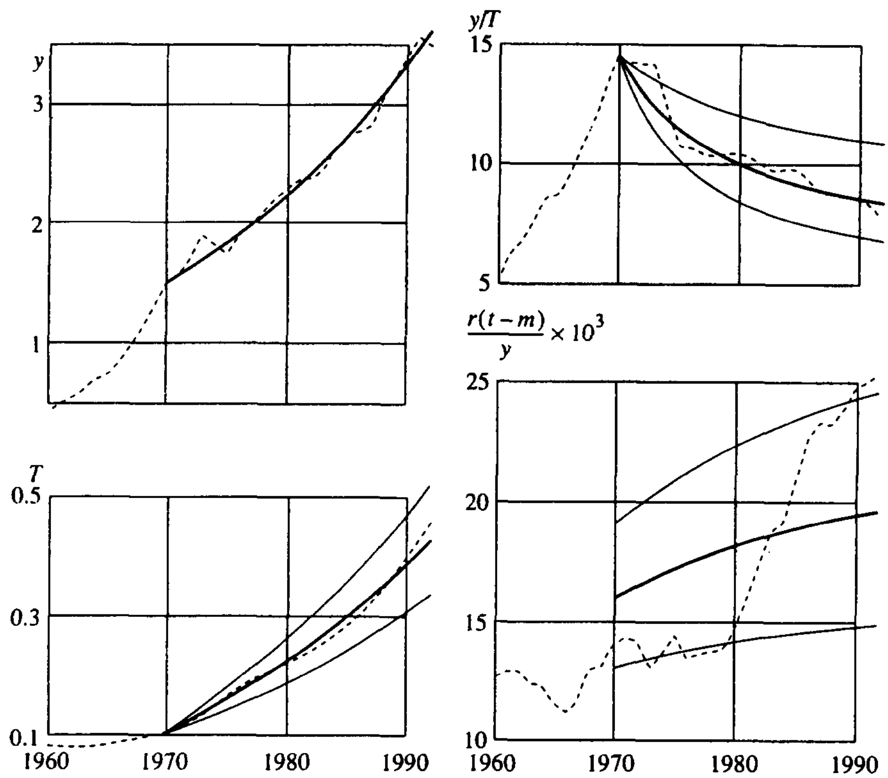

Fig. 2

Experiments showed that, in certain parameter ranges, the optimal trajectories are in good agreement with the econometric data for the Japanese processing industry. This fact confirms the supposition that, during the period from 1970 to 1992, the development of the Japanese economy was close to optimal and is indicative of the adequacy of the mathematical model.

This research was supported financially by the Russian Foundation for Basic Research (99-01-00258, 00-15-96013, 99-01-00146, 00-15-96057, 01-01-06308) and the Fujitsu Research Institute (00-117).

REFERENCES

1. TARASYEV, A. M. and WATANABE, C., Optimal control of *R* and *D* investment in a techno-metabolic system. Interim Report IR-99-001. Intern. Inst. Appl. Syst. Analysis, Laxenburg, Austria, 1999.
2. ARROW, K. J., *Collected Papers*, Vol. 5, *Production and Capital*. Belknap Press, Cambridge, Harvard Univ. Press, 1985.
3. ARROW, K. J. and KURZ, M., *Public Investment, the Rate of Return and Optimal Fiscal Policy*. Baltimore, Johns Hopkins, 1970.
4. GROSSMAN, G. M. and HELPMAN, E., *Innovation and Growth in the Global Economy*. MIT Press, Cambridge, MA, 1993.
5. LEE, C. S. and LEITMANN, G., On one aspect of science policy based on an uncertain model. *Annals of Operat. Res.*, 1999, **88**, 199–214.
6. BORISOV, V. F., HUTSCHENREITER, G. and KRYAZHIMSKII, A. V., Asymptotic growth rates in knowledge-exchanging economies. *Annals of Operat. Res.*, 1999, **89**, 61–73.
7. WATANABE, C., Trends in the substitution of production factors to technology – empirical analysis of the inducing impact of the energy crisis on Japanese industrial technology. *Res. Policy*, 1992, **21**, 481–505.
8. INTRILIGATOR, M., *Mathematical Optimization and Economic Theory*. Prentice-Hall, New York, 1971.
9. GRILICHES, Z., *R and D. Patents, and Productivity*. University of Chicago Press, Chicago, 1987.
10. WATANABE, C., The interaction between technology and economy: national strategies for constrained economic environments. The case of Japan 1955–1922. Interim Report WP-95-16. Intern. Inst. Appl. Syst. Analysis, Laxenburg, Austria, 1995.
11. PONTRYAGIN, L. S., BOLTYANSKII, V. G., GAMKRELIDZE, R. V. and MISHCHENKO, Ye. F., *The Mathematical Theory of Optimal Processes*. Nauka, Moscow, 1983.
12. HARTMAN, *Ph., Ordinary Differential Equations*. Wiley, New York, 1964.
13. PONTRYAGIN, L. S., *Ordinary Differential Equations*. Nauka, Moscow, 1974.

Translated by E.L.S.